

Theorem Pertaining To Some Product of Special Functions

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Abstract: The aim of this paper is to establish a theorem associated with the product of the Fox's H-function, the multivariable H-function and the general class of polynomials. The results of this theorem are unified in nature and producing a very large number of analogous results (new and known) involving simpler special functions and polynomials (of one or more variables) as special cases of our result.

Keywords: H-function, multi variable H-function, general class of Polynomials (Srivastava Polynomial).

1. INTRODUCTION

The series representation of Fox's H-function ([1], [2])

$$H_{P_1, Q_1}^{M_1, N_1} \left[x \left| \begin{matrix} (e_p, E_p) \\ (f_q, F_q) \end{matrix} \right. \right] = \sum_{G=0}^{\infty} \sum_{g=1}^{M_1} (-1)^G \Phi(L_G) x^{L_G} [G! F_g]^{-1},$$

where $\Phi(L_G) = \frac{\prod_{j=1, j \neq G}^{M_1} \Gamma(f_j - F_j L_G) \prod_{j=1}^{N_1} \Gamma(1 - e_j + E_j L_G)}{\prod_{j=M_1+1}^{Q_1} \Gamma(1 - f_j + F_j L_G) \prod_{j=N_1+1}^{P_1} \Gamma(e_j - E_j L_G)}$, $L_G = \frac{(f_g + G)}{F_g}$. (1.1)

The multivariable H-function was defined by H. M. Srivastava and R. Panda [5]

$$H_{p, q}^{0, n; m_1, n_1; \dots; m_r, n_r} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \left| \begin{matrix} (a_j; \alpha_j', \dots, \alpha_j^{(r)})_{1, p} \\ (b_j; \beta_j', \dots, \beta_j^{(r)})_{1, q} \end{matrix} \right. : \begin{matrix} (c_j', \gamma_j')_{1, p_1} \\ \vdots \\ (c_j^{(r)}, \gamma_j^{(r)})_{1, p_r} \end{matrix} \right]$$

$$= \left[\frac{1}{(2\pi\omega)^r} \right] \int_{L_1} \dots \int_{L_r} \phi_1(\xi_1) \dots \phi_r(\xi_r) \psi(\xi_1, \dots, \xi_r) z_1^{\xi_1} \dots z_r^{\xi_r} d\xi_1 \dots d\xi_r,$$
 (1.2)

where $\omega = \sqrt{-1}$,

$$\Phi_i(\xi_i) = \frac{\prod_{j=1}^{m_i} \Gamma(d_j^{(i)} - \delta_j^{(i)} \xi_i) \prod_{j=1}^{n_i} \Gamma(1 - c_j^{(i)} + \gamma_j^{(i)} \xi_i)}{\prod_{j=m_i+1}^{q_i} \Gamma(1 - d_j^{(i)} + \delta_j^{(i)} \xi_i) \prod_{j=n_i+1}^{p_i} \Gamma(c_j^{(i)} - \gamma_j^{(i)} \xi_i)}, \quad i=1, \dots, r$$
 (1.3)

$$\psi(\xi_1, \dots, \xi_r) = \frac{\prod_{j=1}^n \Gamma(1 - a_j + \sum_{i=1}^r \alpha_j^{(i)} \xi_i)}{\prod_{j=n+1}^p \Gamma(a_j - \sum_{i=1}^r \alpha_j^{(i)} \xi_i) \prod_{j=1}^q \Gamma(1 - b_j + \sum_{i=1}^r \beta_j^{(i)} \xi_i)},$$

$$|\arg(z_i)| < \frac{1}{2} \Omega_i \Pi,$$

$$\text{where } \Omega_i = \sum_{j=1}^n \alpha_j^{(i)} - \sum_{j=n+1}^p \alpha_j^{(i)} - \sum_{j=1}^q \beta_j^{(i)} + \sum_{j=1}^{n_i} \gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} \gamma_j^{(i)} + \sum_{j=1}^{m_i} \delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} \delta_j^{(i)} > 0. \quad (1.4)$$

Srivastava has defined and introduced the general polynomials [3]

$$S_{n_1, \dots, n_s}^{m_1, \dots, m_s} = \sum_{k_1=0}^{\lfloor \frac{n_1}{m_1} \rfloor} \dots \sum_{k_s=0}^{\lfloor \frac{n_s}{m_s} \rfloor} \frac{(-n_1)_{m_1 k_1}}{k_1!} \dots \frac{(-n_s)_{m_s k_s}}{k_s!} A[n_1 k_1, \dots, n_s k_s] x_1^{k_1} \dots x_s^{k_s}, \quad (1.5)$$

where $n_i = 0, 1, 2, \dots, \forall i = (1, \dots, s; m_1, \dots, m_s)$ arbitrary positive integers and the coefficients are $A[n_1 k_1, \dots, n_s k_s]$ are arbitrary constants, real or complex. On suitably specializing the coefficients $A[n_1 k_1, \dots, n_s k_s]$, $S_{n_1, \dots, n_s}^{m_1, \dots, m_s}[x_1, \dots, x_s]$ yields a number of known polynomials as its special cases. These include, among others, the Hermite polynomials, the Jacobi polynomials, the Laguerre polynomials, the Bessel's polynomials and several others.

2. MAIN THEOREM

Theorem: Let $\alpha, \beta, \gamma, \lambda, \rho, \xi, u_i, \rho_i, h_j, \theta_j \in \mathbb{R}$, where $(i=1, \dots, s), (j=1, \dots, r)$

$$\text{and if } (1-x)^{\alpha+\beta-\gamma} {}_2F_1[2\alpha, 2\beta; 2\gamma; x] = \sum_{r=0}^{\infty} \beta_r x^r \quad (2.1)$$

then there hold the formula

$$\begin{aligned} & \int_0^1 x^\lambda (x^k+c)^{-\rho} {}_2F_1[\alpha, \beta; \gamma; x] {}_2F_1\left[\gamma-\alpha+\frac{1}{2}, \gamma-\beta+\frac{1}{2}; \gamma+1; x\right] S_{n_1, \dots, n_s}^{m_1, \dots, m_s} \left[c_1 x^{u_1} (x^k+c)^{-\rho_1}, \dots, c_s x^{u_s} (x^k+c)^{-\rho_s} \right] \\ & H_{p, q}^{0, n; m_1, n_1; \dots; m_r, n_r} \left[z_1 x^{h_1} (x^k+c)^{-\theta_1}, z_2 x^{h_2} (x^k+c)^{-\theta_2}, \dots, z_r x^{h_r} (x^k+c)^{-\theta_r} \right] H_{p_1, q_1}^{M_1, N_1} \left[z x^h (x^k+c)^{-\xi} \right] dx \\ & = \sum_{k_1=0}^{\lfloor \frac{n_1}{m_1} \rfloor} \dots \sum_{k_s=0}^{\lfloor \frac{n_s}{m_s} \rfloor} \frac{(-n_1)_{m_1 k_1}}{k_1!} \dots \frac{(-n_s)_{m_s k_s}}{k_s!} A[n_1 k_1, \dots, n_s k_s] \sum_{G=0}^{\infty} \sum_{g=1}^{M_1} (-1)^G \phi(L_G) z^{L_G} [G! F_g]^{-1} \\ & \sum_{r=0}^{\infty} \frac{(\gamma+\frac{1}{2})_r}{(\gamma+1)_r} \beta_r (c_1^{k_1}, \dots, c_s^{k_s}) c^{-(\rho+\xi L_G + \sum_{i=1}^s \rho_i k_i)} H_{p+2, q+2}^{0, n+2; m_1, n_1; \dots; m_r, n_r; 1, 0} \left[\begin{matrix} z_1 c^{-\theta_1} (a_j; \alpha_j', \dots, \alpha_j^{(r)}, 0) \\ z_2 c^{-\theta_2} \\ \vdots \\ z_r c^{-\theta_r} (b_j; \beta_j', \dots, \beta_j^{(r)}, 0) \\ \frac{x^k}{c} \end{matrix} \right] \\ & \left. \begin{aligned} & (1-\rho-\xi L_G - \sum_{i=1}^s \rho_i k_i; \theta_1, \dots, \theta_r, 1) (-\lambda-r-h L_G - \sum_{i=1}^s u_i k_i; h_1, h_2, \dots, h_r, 0) \\ & (1-\rho-\xi L_G - \sum_{i=1}^s \rho_i k_i; \theta_1, \dots, \theta_r, 0) (-1-\lambda-r-h L_G - \sum_{i=1}^s u_i k_i; h_1, h_2, \dots, h_r, 0) \\ & \left(\begin{matrix} (c_j', \gamma_j')_{1, p_1} \\ \dots \\ (c_j^{(r)}, \gamma_j^{(r)})_{1, p_r} \end{matrix} ; - \right) \\ & \left(\begin{matrix} (d_j', \delta_j')_{1, q_1} \\ \dots \\ (d_j^{(r)}, \delta_j^{(r)})_{1, q_r} \end{matrix} ; (0, 1) \right) \end{aligned} \right] \quad (2.2) \end{aligned}$$

provided that:

$$\rho_i > 0, u_i > 0, k_i > 0, (i=1, \dots, s); \theta_j > 0, h_j > 0, (j=1, \dots, r); h > 0, \xi > 0, -\frac{1}{2} < (\gamma-\alpha-\beta) < \frac{1}{2},$$

$$\operatorname{Re} \left(1 + \sum_{i=1}^r h_i \frac{d_i^{(i)}}{\delta_i^{(i)}} \right) > 0 \quad \text{and} \quad |\arg(z_i)| < \frac{1}{2} \Omega_i \Pi, \quad \Omega_i > 0,$$

$$\text{where } \Omega_i = \sum_{j=1}^n \alpha_j^{(i)} - \sum_{j=n+1}^p \alpha_j^{(i)} - \sum_{j=1}^q \beta_j^{(i)} + \sum_{j=1}^{n_i} \gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} \gamma_j^{(i)} + \sum_{j=1}^{m_i} \delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} \delta_j^{(i)} > 0.$$

Proof: We start with Slater result ([7], p.75)

$${}_2F_1[\alpha, \beta; \gamma; x] {}_2F_1\left[\gamma-\alpha+\frac{1}{2}, \gamma-\beta+\frac{1}{2}; \gamma+1; x\right] = \sum_{r=0}^{\infty} \frac{(\gamma+\frac{1}{2})_r}{(\gamma+1)_r} \beta_r x^r, \quad (2.3)$$

where β_r is given by (2.1)

Now, multiplying both sides of (2.3) by

$$\begin{aligned} & x^\lambda (x^k+c)^{-\rho} H_{p, q}^{0, n; m_1, n_1; \dots; m_r, n_r} \left[z x^h (x^k+c)^{-\xi} \right] S_{n_1, \dots, n_s}^{m_1, \dots, m_s} \left[c_1 x^{u_1} (x^k+c)^{-\rho_1}, \dots, c_s x^{u_s} (x^k+c)^{-\rho_s} \right] \\ & H_{p, q}^{0, n; m_1, n_1; \dots; m_r, n_r} \left[z_1 x^{h_1} (x^k+c)^{-\theta_1}, z_2 x^{h_2} (x^k+c)^{-\theta_2}, \dots, z_r x^{h_r} (x^k+c)^{-\theta_r} \right], \text{ integrating with respect to } x \text{ between the limits } 0 \\ & \text{and } 1, \text{ we obtain} \end{aligned}$$

$$\begin{aligned} & \int_0^1 x^\lambda (x^k+c)^{-\rho} {}_2F_1[\alpha, \beta; \gamma; x] {}_2F_1\left[\gamma-\alpha+\frac{1}{2}, \gamma-\beta+\frac{1}{2}; \gamma+1; x\right] S_{n_1, \dots, n_s}^{m_1, \dots, m_s} \left[c_1 x^{u_1} (x^k+c)^{-\rho_1}, \dots, c_s x^{u_s} (x^k+c)^{-\rho_s} \right] \\ & H_{p, q}^{0, n; m_1, n_1; \dots; m_r, n_r} \left[z_1 x^{h_1} (x^k+c)^{-\theta_1}, z_2 x^{h_2} (x^k+c)^{-\theta_2}, \dots, z_r x^{h_r} (x^k+c)^{-\theta_r} \right] H_{p_1, q_1}^{M_1, N_1} \left[z x^h (x^k+c)^{-\xi} \right] dx \end{aligned}$$

$$= \int_0^1 x^\lambda (x^k+c)^{-\rho} \sum_{r=0}^{\infty} \frac{(\gamma+1)_r}{(\gamma+1)_r} \beta_r x^r S_{n_1, \dots, n_s}^{m_1, \dots, m_s} \left[c_1 x^{u_1} (x^k+c)^{-\rho_1}, \dots, c_s x^{u_s} (x^k+c)^{-\rho_s} \right] H_{P_1, Q_1}^{M_1, N_1} \left[z x^h (x^k+c)^{-\xi} \right] \\
 H_{p, q}^{0, n; m_1, n_1; \dots; m_r, n_r} \left[z_1 x^{h_1} (x^k+c)^{-\theta_1}, z_2 x^{h_2} (x^k+c)^{-\theta_2}, \dots, z_r x^{h_r} (x^k+c)^{-\theta_r} \right] dx. \quad (2.4)$$

Interchanging the order of integration and summations which is permissible under the conditions needed in (2.2), we get the following result after a little simplification say (I):

$$I = \sum_{r=0}^{\infty} \frac{(\gamma+1)_r}{(\gamma+1)_r} \beta_r \int_0^1 x^{\lambda+r} (x^k+c)^{-\rho} S_{n_1, \dots, n_s}^{m_1, \dots, m_s} \left[c_1 x^{u_1} (x^k+c)^{-\rho_1}, \dots, c_s x^{u_s} (x^k+c)^{-\rho_s} \right] H_{P_1, Q_1}^{M_1, N_1} \left[z x^h (x^k+c)^{-\xi} \right] \\
 H_{p, q}^{0, n; m_1, n_1; \dots; m_r, n_r} \left[z_1 x^{h_1} (x^k+c)^{-\theta_1}, z_2 x^{h_2} (x^k+c)^{-\theta_2}, \dots, z_r x^{h_r} (x^k+c)^{-\theta_r} \right] dx. \quad (2.5)$$

Using the definitions for general class of polynomials in the series form (1.5), H-function (1.1), and of the multivariable H-function (1.2) on the right of (2.4) and then expressing $(x^k+c)^{-(\rho+\xi L_G + \sum_{i=1}^s \rho_i k_i + \sum_{j=1}^r \theta_j \xi_j)}$ using Srivastava, Goyal [4] and then finally, evaluating the integral on the right hand side with the help of [6], [8] and [9] we arrive at required result after a little simplification.

3. APPLICATIONS AND SPECIAL CASES

The most general nature of multivariable H-function, H-function and general class of polynomials a number of integrals involving simpler functions can be easily evaluated as special cases of the main theorem:

(a) Take $\gamma = \alpha$ in the main theorem, the value of β_r in (2.1) will be equal to $\frac{(\alpha+\frac{1}{2})_r}{r!}$ and the result (2.2) produces the following interesting integral:

$$\int_0^1 x^\lambda (x^k+c)^{-\rho} {}_2F_1 \left[\alpha+\frac{1}{2}, \beta+\frac{1}{2}; \alpha+1; x \right] S_{n_1, \dots, n_s}^{m_1, \dots, m_s} \left[c_1 x^{u_1} (x^k+c)^{-\rho_1}, \dots, c_s x^{u_s} (x^k+c)^{-\rho_s} \right] \\
 H_{p, q}^{0, n; m_1, n_1; \dots; m_r, n_r} \left[z_1 x^{h_1} (x^k+c)^{-\theta_1}, z_2 x^{h_2} (x^k+c)^{-\theta_2}, \dots, z_r x^{h_r} (x^k+c)^{-\theta_r} \right] H_{P_1, Q_1}^{M_1, N_1} \left[z x^h (x^k+c)^{-\xi} \right] dx \\
 = \sum_{k_1=0}^{\lfloor \frac{n_1}{m_1} \rfloor} \dots \sum_{k_s=0}^{\lfloor \frac{n_s}{m_s} \rfloor} \frac{(-n_1)_{m_1 k_1}}{k_1!} \dots \frac{(-n_s)_{m_s k_s}}{k_s!} A[n_1 k_1, \dots, n_s k_s] \sum_{G=0}^{\infty} \sum_{g=1}^{M_1} (-1)^G \varphi(L_G) [G! F_g]^{-1} z^{L_G} \\
 \sum_{r=0}^{\infty} \frac{(\alpha+\frac{1}{2})_r}{(\alpha+1)_r} \frac{(\beta+\frac{1}{2})_r}{r!} (c_1^{k_1}, \dots, c_s^{k_s}) c^{-(\rho+\xi L_G + \sum_{i=1}^s \rho_i k_i)} H_{p+2, q+2}^{0, n+2; m_1, n_1; \dots; m_r, n_r; 1, 0} \left[\begin{matrix} z_1 c^{-\theta_1} \\ z_2 c^{-\theta_2} \\ \vdots \\ z_r c^{-\theta_r} \\ \frac{x^k}{c} \end{matrix} \left(\begin{matrix} a_j; \alpha'_j, \dots, \alpha_j^{(r)}, 0 \\ b_j; \beta'_j, \dots, \beta_j^{(r)}, 0 \end{matrix} \right) \right]; \\
 (1-\rho-\xi L_G - \sum_{i=1}^s \rho_i k_i; \theta_1, \dots, \theta_r, 1) \quad (-\lambda-r-h L_G - \sum_{i=1}^s u_i k_i; h_1, h_2, \dots, h_r, 0) ; \\
 (1-\rho-\xi L_G - \sum_{i=1}^s \rho_i k_i; \theta_1, \dots, \theta_r, 0) \quad (-1-\lambda-r-h L_G - \sum_{i=1}^s u_i k_i; h_1, h_2, \dots, h_r, 0) ; \\
 \left. \begin{matrix} (c'_j, \gamma'_j)_{1, p_1}; \dots; (c'_j, \gamma'_j)_{1, p_r}; - \\ (d'_j, \delta'_j)_{1, q_1}; \dots; (d'_j, \delta'_j)_{1, q_r}; (0, 1) \end{matrix} \right], \quad (3.1)$$

the conditions of validity of (3.1) will follow from those given in (2.2).

(b) Putting $\beta = \alpha + \frac{1}{2}$ then $\alpha + \frac{1}{2} = -v$ (v is non-negative integer) in (3.1), we get

$$\int_0^1 x^\lambda (x^k+c)^{-\rho} (1-x)^v S_{n_1, \dots, n_s}^{m_1, \dots, m_s} \left[c_1 x^{u_1} (x^k+c)^{-\rho_1}, \dots, c_s x^{u_s} (x^k+c)^{-\rho_s} \right] H_{P_1, Q_1}^{M_1, N_1} \left[z x^h (x^k+c)^{-\xi} \right]$$

$$\begin{aligned}
 & H_{p, q; p_1, q_1; \dots; p_r, q_r}^0, n: m_1, n_1; \dots; m_r, n_r \left[z_1 x^{h_1} (x^k+c)^{-\theta_1}, z_2 x^{h_2} (x^k+c)^{-\theta_2}, \dots, z_r x^{h_r} (x^k+c)^{-\theta_r} \right] d x \\
 &= \sum_{k_1=0}^{\lfloor \frac{n_1}{m_1} \rfloor} \dots \sum_{k_s=0}^{\lfloor \frac{n_s}{m_s} \rfloor} \frac{(-n_1)_{m_1 k_1}}{k_1!} \dots \frac{(-n_s)_{m_s k_s}}{k_s!} A[n_1 k_1, \dots, n_s k_s] \sum_{G=0}^{\infty} \sum_{g=1}^{M_1} (-1)^G \varphi(L_G) [G! F_g]^{-1} z^{L_G} \\
 & \sum_{r=0}^{\infty} \frac{(-v)_r}{r!} (c_1^{k_1}, \dots, c_s^{k_s}) c^{-(\rho + \xi L_G + \sum_{i=1}^s \rho_i k_i)} H_{p+2, q+2; p_1, q_1; \dots; p_r, q_r; 0, 1}^0, n+2: m_1, n_1; \dots; m_r, n_r; 1, 0 \\
 & \begin{bmatrix} z_1 c^{-\theta_1} (a_j; \alpha_j', \dots, \alpha_j^{(r)}, 0) ; \\ z_2 c^{-\theta_2} \\ \vdots \\ z_r c^{-\theta_r} (b_j; \beta_j', \dots, \beta_j^{(r)}, 0) ; \\ \frac{x^k}{c} \end{bmatrix} \\
 & (1-\rho-\xi L_G - \sum_{i=1}^s \rho_i k_i; \theta_1, \dots, \theta_r, 1) (-\lambda-r-h L_G - \sum_{i=1}^s u_i k_i; h_1, h_2, \dots, h_r, 0) ; \\
 & (1-\rho-\xi L_G - \sum_{i=1}^s \rho_i k_i; \theta_1, \dots, \theta_r, 0) (-1-\lambda-r-h L_G - \sum_{i=1}^s u_i k_i; h_1, h_2, \dots, h_r, 0) ; \\
 & \left. \begin{array}{l} (c_j', \gamma_j')_{1, p_1} ; \dots ; (c_j^{(r)}, \gamma_j^{(r)})_{1, p_r} ; - \\ (d_j', \delta_j')_{1, q_1} ; \dots ; (d_j^{(r)}, \delta_j^{(r)})_{1, q_r} ; (0, 1) \end{array} \right] , \tag{3.2}
 \end{aligned}$$

the conditions of validity of (3.2) will follow from those given in (2.2)

4. RESULTS AND DISCUSSION

The general nature of H-function, multivariable H-function and the general class of polynomials involve a large variety of polynomials, the main theorem derived in this paper would at once yield a very large number of results, involving a large variety of polynomials and various special functions. Some of the special cases of our theorem have been already discussed here.

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